Finding the N-th digit of Pi

Here is a very interesting formula for pi, discovered by David Bailey, Peter Borwein, and Simon Plouffe in 1995:

$$Pi = SUM_{k=0 \text{ to infinity}} 16^{-k} [4/(8k+1) - 2/(8k+4) - 1/(8k+5) - 1/(8k+6)].$$

The reason this pi formula is so interesting is because it can be used to calculate the N-th digit of Pi (in base 16) *without having to calculate all of the previous digits*!
Moreover, one can even do the calculation in a time that is essentially linear in N, with memory requirements only logarithmic in N. This is far better than previous algorithms for finding the N-th digit of Pi, which required keeping track of all the previous digits!
Presentation Suggestions: You might start off by asking students how they might calculate the 100-th digit of pi using one of the other pi formulas they have learned. Then show them this one...

The Math Behind the Fact: Here's a sketch of how the BBP formula can be used to find the N-th hexadecimal digit of Pi. For simplicity, consider just the first of the sums in the expression, and multiply this by 16^N. We are interested in the fractional part of this expression. The numerator of a given term in this sum is 16^{N-k}, and it can be evaluated very easily mod (8k+1) using a binary algorithm for exponentiation. Division by (8k+1) is straightforward via floating point arithmetic. Not many more than N terms of this sum need be evaluated, since the numerator decreases very quickly as k gets large so that terms become negligible. The other sums in the BBP formula are handled similarly. This yields the hexadecimal expansion of Pi starting at the (N+1)-th digit. More details can be found in the Bailey-Borwein-Plouffe reference.

The BBP formula was discovered using the PSLQ Integer Relation Algorithm. However, the Adamchik-Wagon reference shows how similar relations can be discovered in a way that the proof accompanies the discovery, and gives a 3-term formula for a base 4 analogue of the BBP result.

Sums of Two Squares

Here's a nice theorem due to Fibonacci, in 1202.

Theorem. If integers N and M can each be written as the sum of two squares, so can their product!

Example: since $2=1^2+1^2$ and $34=3^2+5^2$, their product 68 should be expressible as the sum of two squares.

In fact, $68=8^2+2^2$. Is there an easy way to figure out what squares the product will be made of?

Yes! This all follows from the very cool formula:

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2.$$

Presentation Suggestions: Do a few more numerical examples before showing the cool formula.

The Math Behind the Fact: The formula above can be checked trivially. But there are other ways to see why it is true. If you know some linear algebra, take a look at the Fun Fact Really Complex Matrices, and take the determinant of both sides of the matrix equation there. You will get the formula above!

Or, if you know about complex numbers, the left hand side is the squared modulus of two complex numbers and the right side is the squared modulus of their product!

Pi Approximations

Pi is the ratio of the circumference of a circle to its diameter. It is known to be irrational and its decimal expansion therefore does not terminate or repeat. The first 40 places are:

3.14159 26535 89793 23846 26433 83279 50288 41971...

Thus, it is sometimes helpful to have good fractional approximations to Pi. Most people know and use 22/7, since 7*Pi is pretty close to 22. But 22/7 is only good to 2 places. A fraction with a larger denominator offers a better chance of getting a more refined estimate. There is also

333/106, which is good to 5 places.

But an outstanding approximation to Pi is the following:

355/113

This fraction is good to 6 places! In fact, there is no "better approximation" among all fractions (P/Q) with denominators less than 30,000. [By "better approximation" we mean in the sense of how close Q*Pi is to P.]

Presentation Suggestions: Have people verify that 355/113 is a good rational approximation. You can also point out that 355/113 is very easy to remember, since it consists of the digits 113355 in some order!

The Math Behind the Fact: The theory of *continued fractions* allows one to find good rational approximations of any irrational number. This is covered in an introductory course on number theory!

Hairy Ball Theorem

Another fun theorem from topology is the Hairy Ball Theorem. It states that given a ball with hairs all over it, it is impossible to comb the hairs continuously and have all the hairs lay flat. Some hair must be sticking straight up!

A more formal version says that any continuous tangent vector field on the sphere must have a point where the vector is zero.

Is the same true on a torus?

Presentation Suggestions: Draw a picture of a sphere on the board, and have students think together with you how trying to draw a non-zero vector field would cause "problem points", where the field is not continuous.

The Math Behind the Fact: If you've done the Fun Fact on the Euler characteristic, students will find it very surprising that the number of "problem points" of a vector field on a surface is related to the Euler characteristic of that surface! Namely, every point has an "index" that describes how many times the vector field rotates in a neighborhood of the problem point. The sum of the indices of all the vector fields will be the Euler characteristic. Since the torus has Euler number 0, it is possible to have a vector field on it without any "problem points".

Proofs without Words

There are many facts which can be proved just by looking at a picture! For instance:

$$x^2 - y^2 = (x+y)\,(x-y)$$

can be illustrated by drawing a large square, cutting a small square out of the corner, and cutting off the rectangular portion at the top and placing on the side to illustrate the right hand side of this identity.

Presentation Suggestions: Write down the identity and then draw the picture, asking students to see if they can figure out how the picture explains the identity. Or, draw the picture and ask if they can figure out what it should prove!

The Math Behind the Fact: Almost any "Proof without Words" from Mathematics Magazine or Nelsen's recent book (see reference) will serve as a short, snappy Fun Fact!

Fourier Ears Only

Did you know that every sufficiently smooth function on an interval can be expressed as an infinite sum of sines and cosines of various frequencies and amplitudes?

Yes, it's true! It is analogous to the fact that you can approximate irrational numbers as the sum of a bunch of rational numbers (that's what a decimal expansion of an irrational number *is*, after all)! Writing a function as a sum of sines and cosines is called a *Fourier series*.

In fact, your ears do Fourier series automatically! There are little hairs (*cilia*) in you ears which vibrate at specific (and different) frequencies. When a wave enters your ear, the cilia will vibrate if the wavefunction "contains" any component of the correponding frequency! Because of this, you can distinguish sounds of various pitches!

Presentation Suggestions: This is a great Fun Fact to reinforce the connection of mathematics with other disciplines. You can show students how to find Fourier series by working examples into a homework on integration.

The Math Behind the Fact: You can learn about Fourier series in an advanced differential equations course, one which covers boundary value problems, or an advanced course in analysis. Fourier series are used to find solutions to partial differential equations, such as problems involving heat flow. Fourier series can be used to construct some pathological functions such as one which is continuous but nowhere differentiable.

By the way, one type of "sufficiently smooth" function (as mentioned in the first sentence above) is a function that is piecewise differentiable. Continuous is not enough; see the reference.

Rental Harmony

You and your college friends decide to rent a house together, and the N of you have found a house with N bedrooms. However, the house has rooms of different sizes, different features, and each of you have different preferences.

Is it always possible to split the rent and price the rooms in such a way that each person will want a different room?

The answer is yes, under mild conditions! The *Rental Harmony Theorem* (Su, 1999) says:

If the following conditions hold:

- (Good House) Each player finds some room acceptable in every pricing scheme,
- (Closed Preference Sets) A room that is preferred for a convergent sequence of prices will be preferred in the limit,
- (Miserly Tenants) In any pricing scheme which includes a free room, the most expensive room is never chosen,

then there will be a pricing scheme in which each person will prefer a different room!

By the way, if you drop the Miserly Tenants condition, the theorem is still true if you allow "negative rents", i.e., you can still find a solution but it may be one in which you are paying one of the other housemates to live with you! (In that case you could ditch the subsidized housemate and use the extra room and extra money in other ways?)

Medical Tests and Bayes' Theorem

Suppose that you are worried that you might have a rare disease. You decide to get tested, and suppose that the testing methods for this disease are accurate 99 percent of the time (regardless of whether the results come back positive or negative). Suppose this disease is actually quite rare, occurring randomly in the general population in only one of every 10,000 people.

If your test results come back positive, what are your chances that you actually have the disease?

Do you think it is approximately: (a) .99, (b) .90, (c) .10, or (d) .01?

Surprisingly, the answer is (d), less than 1 percent chance that you have the disease!

Presentation Suggestions: After discussing the reasons why the test results are not so reliable, see how changing the parameters affects the outcome. Would the result be so surprising if the disease were more common? How would things change if you allow the percentage of false positives and false negatives to be different?

The Math Behind the Fact: This fact may be deduced using something called Bayes' theorem, a computational device used to find the probability of event A given event B, written P(A|B), in terms of the probability of B given A, written P(B|A), and the probabilities of A and B:

$$P(A|B)=P(A)P(B|A) / P(B)$$

In this case, event A is the event you have this disease, and event B is the event that you test positive. Here, P(B|A)=.99, P(A)=.0001, and P(B) may be derived by conditioning on whether event A does or does not occur:
P(B)=P(B|A)P(A)+P(B|not A)P(not A)
or .99*.0001+.01*.9999, which is less than 1 percent.
Alternatively, we can see this by thinking about what we can expect in 1 million cases. In those million, about 100 will have the disease, and on average 99 of those cases will be correctly diagnosed as having it. Otherwise about 999,900 of the million will not have the disease, but of

those cases 9999 of those will be false positives (test results that are positive because of errors). So, if you test positive, then the likelihood that you actually have the disease is 99/(99+9999), which gives the same fraction as above, approximately .0098 or less than 1 percent!
You may wish to discuss why these calculations wouldn't hold if the disease were not independently and identically distributed throughout the population. For instance, if the disease were a cancer like breast cancer that runs in families or mesothelioma related to asbestos exposure in the workplace, then the estimate of P(A)=.0001 would not be correct.

One, Two, Three, Pi

Here's an interesting formula:

$$\text{Arctan}(1) + \text{Arctan}(2) + \text{Arctan}(3) = \text{Pi}.$$

(Everything's in radians, of course).
Presentation Suggestions: Challenge students to prove this fact.

The Math Behind the Fact: The Figure contains a hint; check that the angles of the three triangles at their common vertex add to Pi. Can you find a similar proof for the following equation?

Arctan(1/2)+Arctan(1/3) = Pi/4.

Morley's Theorem

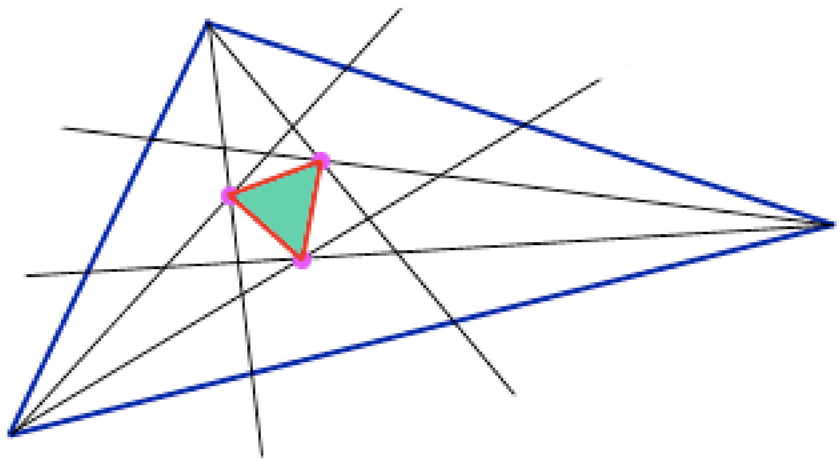

Figure 1

Take any triangle. Mark the 3 points which are the intersections of adjacent angle trisectors. No matter what triangle you start with, these 3 points will form an equilateral triangle!

Presentation Suggestions: Draw a picture and invite students to draw their own triangle on a sheet of paper and follow along.

The Math Behind the Fact: This is yet another surprising fact from geometry. Can you think of a proof?

Multiplicat

If you liked the Fun Fact Multiplication by 11, you'll enjoy seeing how to take that idea one step farther. Here's a quick way to multiply by 111.

To multiply a two-digit number by 111, add the two digits and if the sum is a single digit, write this digit TWO TIMES in between the original digits of the number. Some examples:

23x111= 2553
41x111= 4551

The same idea works if the sum of the two digits is not a single digit, but you should write down the last digit of the sum twice, but remember to carry if needed. So

57x111= 6321

because 5+7=12, but then you have to carry the one twice.
If the number you are multiplying by 111 is a three-digit number, say ABC, then the answer will have five digits (though it may be six if there is a carry involved): the first digit is A, the second digit is A+B, the third digit is A+B+C,

the fourth digit is B+C, the fifth digit is C.

Again, you must remember to carry if any of these sums is more than one digit. Thus 123x111=13653, 241x111=26751, and for an example where carrying is needed, 352x111=39072. (Because of the carries, it may be easier to do the sums and write the answer down from right to left.)

Presentation Suggestions: Do the Multiplication by 11 Fun Fact first.

The Math Behind the Fact: Multiply using the traditional (long) method for multiplication, and you will find that the above shortcut works because it is doing exactly the same sums that you would have to do using the traditional method for multiplication.

Spherical Geometry

Remember high school geometry? The

sum of the angles of a planar triangle is always 180 degrees or Pi radians. However, triangles on other surfaces can behave differently!

For instance, consider a triangle on a sphere, whose edges are "intrinsically" straight in the sense that if you were a very tiny ant living on the sphere you would not think the edges were bending either to the left or right. (Such intrinsically straight lines are called *geodesics*. On spheres, they correspond to pieces of great circles whose center coincide with the center of the sphere.)

A triangle on a sphere has the interesting property that the sum of the angles is *greater than 180* degrees! And in fact, two triangles with the same angles are not just similar (as in planar geometry), they are actually *congruent*! But wait, there's more: on a UNIT sphere, the AREA of the triangle actually satisfies:

AREA of a triangle = (sum of angles) - Pi ,

where the angles are measured in radians. Cool!
Another neat fact about spherical triangles may be found in Spherical Pythagorean Theorem.
Presentation Suggestions: Demonstrate the assertions about angles and areas with an example: draw a picture of a sphere and then draw a triangle whose vertices are at the north pole and at two distinct points on the equator. Here's a follow-up question for your students: are geodesic paths always the shortest paths between two points?

The Math Behind the Fact: Planar

geometry is sometimes called *flat* or *Euclidean* geometry. The geometry on a sphere is an example of a *spherical* or *elliptic* geometry. Another kind of non-Euclidean geometry is hyperbolic geometry. Spherical and hyperbolic geometries do not satify the parallel postulate.

By the way, 3-dimensional spaces can also have strange geometries. Our universe, for instance, seems to have a Euclidean geometry on a local scale, but does not on a global scale. In much the same way that a sphere is "curved", so that divergent geodesics extending from the south pole will meet again at the north pole, Einstein suggested that 3-space is "curved" by the presence of matter, so that light rays (which follow geodesics) bend near very massive objects!

Red-Black

Here's a pretty easy card trick that you can do that can also be pretty surprising. Here's how the trick you do will appear to others:

Take a deck of cards, and give it to a spectator and ask her to shuffle the deck and return it to you face down. You take the cards, and (with a little showmanship but without looking at the fronts of the cards) separate them into two piles, and then say "Just by feeling the redness or blackness of the cards with my fingers, I've made two piles so that the number of red cards in the first pile is the number of black cards in the second pile."

Have your spectator turn over the cards and verify!

Presentation Suggestions: Your spectator can shuffle the cards as many times as she likes--- it won't matter! When she gives the cards to you, all you are really doing (though don't make it obvious) is counting the cards into two piles so that there are 26 cards in each pile.

The Math Behind the Fact: The reason this trick works is simple... if the number of red cards in the first and second piles is R and S, and the number of black cards in the first and second piles is A and B, then we know that R+S=26 (since the total number of red cards is 26) and S+B=26 (since the total number of cards in the second pile is 26). These two equations can be subtracted from one another to show that R-B=0, or R=B.

For more fun, try the Red-Black Pairs Card Trick, or one of the other Fun Facts on mathematical magic.

Take a any sphere and slice it up by parallel knives 1 inch apart. The areas of the spherical bands between the cuts (on the surface of the sphere) are exactly the same area!

Presentation Suggestions: Draw a suggestive picture!

The Math Behind the Fact: This begins to make sense when you reflect on it. The bands are wider near the top of the sphere because it is more curved there, but the diameter is smaller there. These effects cancel exactly! It is makes a nice homework exercise in multivariable calculus.

Pizza Slices

Take a pizza and pick an arbitrary point in it. Suppose you cut the pizza into 8 slices by cutting at 45 degree angles through that point, and color the alternate pieces red and green.

Surprising theorem: the total area of the red slices and the total area of the green slices will always be the same!

In fact, this theorem is true if the number of slices is any multiple of 4 except for 4, and the slices are cut by using equal angles through a fixed arbitrary point in the pizza.

Alternatively, if instead of equal angles, you use equal-length arcs on the circumference and slice from a fixed arbitrary point in the pizza, the conclusion still holds if the number of slices is even and greater than 2.

Presentation Suggestions: Draw a few pictures to illustrate some special cases.

The Math Behind the Fact: The theorem can be proved by using calculus and polar coordinates. The reference gives background and generalizations.

one of my favorite theorems from topology, called the Ham Sandwich Theorem.

It says: given globs of ham, bread, and cheese (in any shape), placed any way you like, there exists one flat slice of a knife (a plane) that will bisect each of the ham, bread, and cheese.

In other words you can share it with a friend so that both of you get exactly the same amounts of all three globs!

Moreover, the ham, bread, and cheese don't even need to be near each other, nor do they have to be connected!

Presentation Suggestions: Get your class to envision various scenarios for the location of the ham, bread, and cheese, and see if they can figure out where the plane should go. By the way, a common error is to think that the plane should pass through the centers of mass of the three objects. But this is not necessarily the case, as the reader can check by constructing some simple examples using lopsided volumes.

The Math Behind the Fact: There is a version that holds in N-dimensional space, that says any N globs of positive volume can be simultaneously bisected by a single hyperplane. Like the Brouwer fixed point theorem and the Borsuk-Ulam theorem, this has an existence proof... it doesn't say where the plane is! Actually, the Ham Sandwich Theorem can be proved using the Borsuk-Ulam theorem.

Squaring Quickly

You may have seen the Fun Fact on squares ending in 5; Here's a trick that can help you square ANY number quickly.

It's based on the algebra identity for the difference of squares, but with a twist! Can you figure it out?

$$54^2 = 50 * 58 + 4^2 = 2916.$$
$$42^2 = 40 * 44 + 2^2 = 1764.$$
$$37^2 = 34 * 40 + 3^2 = 1369.$$

You have to pretty proficient at multiplying one digit numbers by two digit numbers in your head to do this trick well. But if you master this, then you can build upon it in some amazing ways: $116^2 = 100 * 132 + 16^2 = 13{,}200 + 256 = 13{,}456.$ Thinking CREATIVELY about everything you learn, no matter how trivial it may seem, will allow you to find some really clever applications!

Presentation Suggestions: If you practice this a LOT beforehand, you can start off by asking students to name any 2-digit number and you will do it in your head quickly. Then tell them the trick. But only do this with a LOT OF PRACTICE!

The Math Behind the Fact: If you look closely, we are using the identity:

$$a^2 = (a-b)(a+b) + b^2.$$

The reference contains more ideas for doing fast mental calculations. See also Fun Facts on lightning arithmetic.

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Koch Snowflake

Snowflakes are amazing creations of nature. They seem to have intricate detail no matter how closely you look at them. One way to model a snowflake is to use a *fractal* which is any mathematical object showing "self-similarity" at all levels.

The Koch snowflake is constructed as follows. Start with a line segment. Divide it into 3 equal parts. Erase the middle part and substitute it by the top part of an equilateral triangle. Now, repeat this procedure for each of the 4 segments of this second stage. See Figure 1. If you continue repeating this procedure, the curve will never self-intersect, and in the limit you get a shape known as the *Koch snowflake*.

Amazingly, the Koch snowflake is a curve of infinite length!

And, if you start with an equilateral triangle and do this procedure to each side, you will get a snowflake, which has finite area, though infinite boundary!

Presentation Suggestions: Draw pictures. If they like this Fun Fact, ask them: can you figure out how to construct a 3-dimensional example? [Hint: start with a regular tetrahedron. See Koch Tetrahedron for what happens.]

The Math Behind the Fact: You can see that the boundary of the snowflake has infinite length by looking at the lengths at each stage of the process, which grows by 4/3 each time the process is repeated. On the other hand, the area inside the snowflake grows like an infinite series, which is geometric and converges to a finite area! You can learn about fractals in a course on dynamical systems.

Scrabble Identity

Question: What positive integer, when spelled out, has a Scrabble score equal to that integer?

Answer: see Figure 1.

Euler's Formula

The five most important numbers in mathematics all appear in a single equation!

$$e^{i*\mathrm{Pi}} + 1 = 0.$$

In fact, this is a special case of the following formula, due to Euler:

$$e^{it} = \cos(t) + i*\sin(t).$$

Presentation Suggestions: This is a good Fun Fact to use after introducing complex numbers, as it gives some intuition about polar coordinates on C. However, a more interesting use is after teaching the Taylor series of *e*, sin , and cos. See below. You could do the following in class or on a Taylor series homework and then give the Fun Fact as the case where you set t=Pi.

The Math Behind the Fact: Introduce the "imaginary number" *i*, a number with the property that i^2=-1. Make sure students understand that, say, i^5=*i*. Take the Taylor series of e^t and plug "it" in (that's "*i**t"). Since e^t converges absolutely everywhere, have them rearrange the resulting series into two series: one with an i in each term, and one with no i's. What are these two series? Yes, cos(t) and i*sin(t).

This formula demonstrates a remarkable connection between analysis (in the form of the Taylor series of e, sin, and cos) and geometry (the polar coordinates in C). Heck, it's a remarkable connection between e, sin and cos!

Rolling Polygons

Perhaps you've learned from a calculus class that as you roll a circular disk along a straight line, that the area under the cycloid swept out by following a point on the edge of the disk between two successive points of tangency is exactly 3 times the area of the disk.

But did you know that a very similar fact is true for polygons?

For instance, take a square on a flat line, and mark one corner on the line with a red dot. Now "roll" it along the line by pivoting the square around the corner that touches the line. Each time it comes to a rest, mark the position of the red dot. When the red dot again touches the line, stop.

Connect the red dots with *straight lines*. (These are dotted lines in the Figure.) The area under this polygonal region will be 3 times the area of the square. You can verify this in Figure 1.

The same holds for pentagons, hexagons, and any regular n-gon!

Presentation Suggestions: Draw examples on the board! Challenge students to show this fact true for a triangle or a pentagon (harder).

The Math Behind the Fact: Regular n-gons with a large number of sides are approximately circular, and the polygonal path obtained by connecting the dots will approximately converge to the path taken by a point on the edge of the disk! This recovers the result for the cycloid.

Sums of Two Squares,

Which whole numbers are expressible as sums of two (integer) squares?

Here's a theorem that completely answers the question, due to Fermat: A number N is expressible as a sum of 2 squares if and only if in the prime factorization of N, every prime of the form (4k+3) occurs an even number of times!

Examples: 245 = 5*7*7. The only prime of the form 4k+3 is 7, and it appears twice. So it should be possible to write 245 as a sum of 2 squares (in fact, try the squares of 14 and 7). But because 7 appears only once in 42=2*3*7, it is impossible to write 42 as the sum of two squares.

A corollary of this fact is that every prime of the form (4k+1) can be written as the sum of two squares.

Presentation Suggestions: See if your students can figure out how to write 245 as the sum of two squares, using their knowledge from the Fun Fact Sums of Two Squares and writing 5, and 49 as the sum of two squares.

The Math Behind the Fact: Well, if not every number can be written as a sum of two squares, perhaps a larger number of squares will suffice? Can every number be written as the sum of 3 or 4 squares? See the Fun Fact Sums of Three and Four Squares for more on these questions.